ELSEVIER
Journal of Computational and Applied Mathematics 132 (2001) 247–253
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.nl/locate/cam

Positive solution of a singular nonlinear third-order periodic boundary value problem

Lingbin Kong[a,*], Shoutian Wang[a], Junyu Wang[b]

[a] *Department of Mathematics, Daqing Petroleum Institute, Heilongjiang, Anda 151400, People's Republic of China*

[b] *Department of Mathematics, Jilin University, Changchun 130023, People's Republic of China*

Received 2 August 1999; received in revised form 30 January 2000

Abstract

The singular nonlinear third-order periodic boundary value problem $u''' + \rho^3 u = f(t,u)$, $0 \leqslant t \leqslant 2\pi$, with $u^{(i)}(0) = u^{(i)}(2\pi)$, $i = 0,1,2$, is studied, where $\rho > 0$ and f is singular at $u = 0$. Under suitable conditions on f, it is proved by employing a priori estimates, perturbation technique and Schauder fixed point theorem that the problem has at least one positive solution if $\rho \in (0, \frac{1}{\sqrt{3}})$. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 34B15

Keywords: Third-order problem; A priori estimates; Perturbation technique

1. Introduction

The nonlinear periodic boundary value problems have been widely studied by a number of authors in recent years. For details, see [1–13] and references therein. However, the boundary value problems treated in the above-mentioned references are not able to possess singularity, and singular nonlinear periodic boundary value problems are rarely considered. In recent paper, Wang and Jiang [14] have established the existence and uniqueness of results for the singular nonlinear second-order periodic boundary value problem

$$\begin{aligned} &-u'' + \rho^2 u = f(t,u), \quad 0 \leqslant t \leqslant 2\pi, \\ &u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{aligned} \tag{1.1}$$

where $\rho > 0$ and f may be singular at $u = 0$. There, a priori estimates are used as a fundamental tool to conclude the existence of positive solution of problem (1.1).

* Corresponding author.

PII: S0377-0427(00)00325-3

Motivated by the results of (1.1), in this paper we study the singular nonlinear third-order periodic boundary value problem

$$u''' + \rho^3 u = f(t,u), \quad 0 \leqslant t \leqslant 2\pi,$$
$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0,1,2, \tag{1.2}$$

where ρ is a positive constant.

Our hypotheses are as follows.

(H_1) $f(t,u)$ is a nonnegative function defined on $[0,2\pi] \times (0,+\infty)$, and $f(t,u)$ is integral on $[0,2\pi]$ for each fixed $u \in (0,+\infty)$;

(H_2) $f(t,u)$ is nonincreasing in $u > 0$ for almost all $t \in [0,2\pi]$ and

$$\lim_{u\to 0^+} f(t,u) = +\infty, \quad \lim_{u\to+\infty} f(t,u) = 0$$

hold uniformly for $t \in [0,2\pi]$;

(H_3) For each fixed constant $\theta > 0$, inequality $\int_0^{2\pi} f(s,\theta)\,\mathrm{d}s < +\infty$ holds.

A function $u(t)$ is said to be a positive solution to problem (1.2), if it satisfies

(i) $u \in C^2[0,2\pi]$, $u^{(i)}(0) = u^{(i)}(2\pi)$, $i = 0,1,2$;
(ii) u''' exists almost everywhere and $u(t) > 0$ in $[0,2\pi]$ and

$$u'''(t) + \rho^3 u(t) = f(t,u(t)) \quad \text{a.e. on } [0,2\pi].$$

The main result of this paper is as follows.

Theorem 1. *Assume that* (H_1)–(H_3) *hold. Then problem* (1.2) *has at least one positive solution if* $\rho \in (0, \frac{1}{\sqrt{3}})$.

2. Proof of Theorem 1.

Lemma 1. *If* $\rho \in (0,+\infty)$, *then the linear problem*

$$u'' - \rho u' + \rho^2 u = 0,$$
$$u(0) - u(2\pi) = 0, \quad u'(0) - u'(2\pi) = 1 \tag{2.1}$$

has a unique positive solution

$$w(t) = \frac{2\mathrm{e}^{(\rho/2)t}\left[\sin\dfrac{\sqrt{3}}{2}\rho(2\pi - t) + \mathrm{e}^{-\rho\pi}\sin\dfrac{\sqrt{3}}{2}\rho t\right]}{\sqrt{3}\rho(\mathrm{e}^{\rho\pi} + \mathrm{e}^{-\rho\pi} - 2\cos\sqrt{3}\rho\pi)}. \tag{2.2}$$

Proof. By a direct calculation, we can easily prove Lemma 1.

For every function $u \in C[0,2\pi]$, we define the operator

$$(Ju)(t) := \int_0^{2\pi} g(t,x)u(x)\,\mathrm{d}x, \tag{2.3}$$

where

$$g(t,x) := \begin{cases} \dfrac{e^{\rho(2\pi+x-t)}}{e^{2\rho\pi}-1}, & 0 \leqslant x \leqslant t \leqslant 2\pi, \\ \dfrac{e^{\rho(x-t)}}{e^{2\rho\pi}-1}, & 0 \leqslant t \leqslant x \leqslant 2\pi. \end{cases} \tag{2.4}$$

Now, we consider the problem

$$u'' - \rho u' + \rho^2 u = f(t, Ju),$$

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1 \tag{2.5}$$

for which we have the following result.

Lemma 2. *Let $w(t)$ be a unique solution of* (2.1), *then problem* (2.5) *is equivalent to integral equation*

$$u(t) = \int_0^{2\pi} G(t,s) f(s, (Ju)(s))\,\mathrm{d}s, \tag{2.6}$$

where

$$G(t,s) = \begin{cases} w(t-s), & 0 \leqslant s \leqslant t \leqslant 2\pi, \\ w(2\pi + t - s), & 0 \leqslant t \leqslant s \leqslant 2\pi \end{cases}$$

$$= \begin{cases} \dfrac{2e^{(\rho/2)(t-s)}[\sin \frac{\sqrt{3}}{2}\rho(2\pi - t + s) + e^{-\rho\pi} \sin \frac{\sqrt{3}}{2}\rho(t-s)]}{\sqrt{3}\rho(e^{\rho\pi} + e^{-\rho\pi} - 2\cos\sqrt{3}\rho\pi)}, & s \leqslant t, \\ \dfrac{2e^{(\rho/2)(2\pi+t-s)}[\sin \frac{\sqrt{3}}{2}\rho(s-t) + e^{-\rho\pi} \sin \frac{\sqrt{3}}{2}\rho(2\pi - s + t)]}{\sqrt{3}\rho(e^{\rho\pi} + e^{-\rho\pi} - 2\cos\sqrt{3}\rho\pi)}, & s \geqslant t. \end{cases} \tag{2.7}$$

Proof. As shown in [10], by (2.6) it is easily vertified that

$$u'(t) = \int_0^t w'(t-s) f(s, (Ju)(s))\,\mathrm{d}s + \int_t^{2\pi} w'(2\pi + t - s) f(s, (Ju)(s))\,\mathrm{d}s,$$

$$u''(t) = \int_0^t w''(t-s) f(s, (Ju)(s))\,\mathrm{d}s + \int_t^{2\pi} w''(2\pi + t - s) f(s, (Ju)(s))\,\mathrm{d}s + f(t, (Ju)(t))$$

and hence we can prove Lemma 2. □

Lemma 3. *Let $\rho \in (0, \frac{1}{\sqrt{3}})$, then we have the estimates*

$$\frac{2\sin\sqrt{3}\rho\pi}{\sqrt{3}\rho(e^{\rho\pi}+1)^2} \leqslant G(t,s) \leqslant \frac{2}{\sqrt{3}\rho\sin\sqrt{3}\rho\pi}, \quad t, s \in [0, 2\pi]. \tag{2.8}$$

Proof. If $x = t/2$ and $h(x) = \sin\sqrt{3}\rho(\pi - x) + e^{-\rho\pi}\sin\sqrt{3}\rho x$, we have

$$w(t) = \frac{2e^{\rho x} h(x)}{\sqrt{3}\rho(e^{\rho\pi} + e^{-\rho\pi} - 2\cos\sqrt{3}\rho\pi)}. \tag{2.9}$$

If $e^{-\rho\pi} \leqslant \cos\sqrt{3}\rho\pi$, then by a direct computation, we get $h''(x) < 0$ in $[0,\pi]$ and $h'(0) \leqslant 0$, and hence $h'(x) \leqslant 0$ in $[0,\pi]$. Thus, $h(x)$ is nonincreasing in $[0,\pi]$; moreover, we have

$$e^{-\rho\pi}\sin\sqrt{3}\rho\pi = h(\pi) \leqslant h(x) \leqslant h(0) = \sin\sqrt{3}\rho\pi, \quad x \in [0,\pi]. \tag{2.10}$$

If $e^{-\rho\pi} > \cos\sqrt{3}\rho\pi$, since $h''(x) < 0$ in $[0,2\pi]$, then $h(x)$ is a concave function in $[0,2\pi]$. It follows from $h(0) > h(\pi)$ and $h'(0) > 0$ that there exists a $x_0 \in (0,\pi)$ such that $h(x_0) = \max_{x\in[0,\pi]} h(x)$. It is easy to know that the critical point of h is given by

$$x = \frac{1}{\sqrt{3}\rho}\left[\operatorname{arctg}\left(\frac{e^{-\rho\pi} - \cos\sqrt{3}\rho\pi}{\sin\sqrt{3}\rho\pi}\right) + k\pi\right],$$

where $k = 0, \pm 1, \pm 2, \ldots$.

If $k < 0$, since

$$0 < \operatorname{arctg}\left(\frac{e^{-\rho\pi} - \cos\sqrt{3}\rho\pi}{\sin\sqrt{3}\rho\pi}\right) < \frac{\pi}{2},$$

we get $x < 0$, which is a contradiction. Moreover, it is easy to know $k = 0$ by the concavity of $h(x)$ in $[0,\pi]$. So we have

$$x_0 = \frac{1}{\sqrt{3}\rho}\operatorname{arctg}\left(\frac{e^{-\rho\pi} - \cos\sqrt{3}\rho\pi}{\sin\sqrt{3}\rho\pi}\right) \tag{2.11}$$

and, hence,

$$\begin{aligned} h(x_0) &= \sin\sqrt{3}\rho(\pi - x_0) + e^{-\rho\pi}\sin\sqrt{3}\rho x_0 \\ &= \frac{\cos\sqrt{3}\rho\pi}{\sqrt{1+\operatorname{tg}^2\sqrt{3}\rho x_0}}\left[\operatorname{tg}\sqrt{3}\rho\pi + \left(\frac{e^{-\rho\pi} - \cos\sqrt{3}\rho\pi}{\sin\sqrt{3}\rho\pi}\right)\operatorname{tg}\sqrt{3}\rho x_0\right] \\ &= \sqrt{1 - 2e^{-\rho\pi}\cos\sqrt{3}\rho\pi + e^{-2\rho\pi}}. \end{aligned} \tag{2.12}$$

By (2.9), (2.10) and (2.12), we obtain

$$\frac{2\sin\sqrt{3}\rho\pi}{\sqrt{3}\rho(e^{\rho\pi}+1)^2} \leqslant w(t) \leqslant \frac{2}{\sqrt{3}\rho\sin\sqrt{3}\rho\pi},$$

and so does $G(t,s)$. The proof is complete. □

Lemma 4. *Let* $\rho \in (0, \frac{1}{\sqrt{3}})$. *If* $u(t)$ *is any positive solution to the problem* (2.5), *then there exist two constants* $0 < r < R$ *such that* $r \leqslant u(t) \leqslant R$ *in* $[0,2\pi]$.

Proof. Let $u(t)$ be a solution to problem (2.5) and let $r = \min_{t\in[0,2\pi]} u(t)$ and $R = \max_{t\in[0,2\pi]} u(t)$, then by (2.6) and (2.8), we get

$$\frac{2\sin\sqrt{3}\rho\pi}{\sqrt{3}\rho(e^{\rho\pi}+1)^2}\int_0^{2\pi} f(s,(Ju)(s))\,ds \leqslant r \leqslant R \leqslant \frac{2}{\sqrt{3}\rho\sin\sqrt{3}\rho\pi}\int_0^{2\pi} f(s,(Ju)(s))\,ds.$$

Consequently, we have

$$R \leqslant r \left(\frac{e^{\rho\pi} + 1}{\sin \sqrt{3}\rho\pi} \right)^2.$$

If $u(t) \geqslant r$ is not true, then there exists a sequence of positive solution to (2.5), $\{u_j(t)\}_{j=1}^{\infty}$, such that

$$r_j = \min_{t\in[0,2\pi]} u_j(t) \to 0 (j \to \infty).$$

On the other hand, by (H_2), (2.3) and (2.8), we have

$$\begin{aligned} r_j &= \min_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s) f(s,(Ju_j)(s))\,ds \\ &\geqslant \frac{2\sin\sqrt{3}\rho\pi}{\sqrt{3}\rho(e^{\rho\pi}+1)^2} \int_0^{2\pi} f\left(s, \frac{2\pi e^{2\rho\pi}}{e^{2\rho\pi}-1} R_j\right) ds \\ &\geqslant \frac{2\sin\sqrt{3}\rho\pi}{\sqrt{3}\rho(e^{\rho\pi}+1)^2} \int_0^{2\pi} f\left(s, \frac{2\pi(e^{\rho\pi}+1)^2}{(1-e^{-2\rho\pi})\sin\sqrt{3}\rho\pi} r_j\right) ds \\ &\to +\infty (j \to \infty), \end{aligned}$$

which is a contradiction.

If $u(t) \leqslant R$ is not true, then there exists a sequence of positive solution to (2.5), $\{u_j(t)\}_{j=1}^{\infty}$, such that

$$R_j = \max_{t\in[0,2\pi]} u_j(t) \to +\infty \quad (j \to \infty).$$

In addition, by (H_2) we have

$$\begin{aligned} R_j &= \max_{t\in[0,2\pi]} \int_0^{2\pi} G(t,s) f(s,(Ju_j)(s))\,ds \\ &\leqslant \frac{2}{\sqrt{3}\rho\sin\sqrt{3}\rho\pi} \int_0^{2\pi} f\left(s, \frac{2\pi r}{e^{2\rho\pi}-1}\right) ds < +\infty, \end{aligned}$$

which is also a contradiction.

Now we prove Theorem 1. Let

$$K := \{u \in C[0,2\pi];\, u(t) \geqslant 0 \text{ on } [0,2\pi]\},$$

then K is a normal cone in $C[0,2\pi]$.

To obtain a solution of (2.6), we seek a fixed point of the integral operator

$$(\Phi u)(t) := \int_0^{2\pi} G(t,s) f(s,(Ju)(s))\,ds,$$

where $G(t,s)$ is given by (2.7). Due to the singularity of f given by (H_2), Φ is not defined on all of the cone K.

Next, for each $n \geqslant 1$, we define a sequence of function f_n by

$$f_n(t,(Ju)(t)) = f\left(t, \max\left\{(Ju)(t), \frac{1}{n}\right\}\right),$$

then f_n is continuous on $[0,2\pi]\times[0,+\infty)$ and nonincreasing in $Ju > 0$ for all $t \in [0,2\pi]$. Furthermore,

$$f_n(t,(Ju)(t)) \leqslant f(t,(Ju)(t)), \quad f_n(t,(Ju)(t)) \leqslant f\left(t, \frac{1}{n}\right).$$

Now, we define a sequence of operator $\Phi_n : K \to K$, $n \geqslant 1$, by

$$(\Phi_n u)(t) := \int_0^{2\pi} G(t,s) f_n(s,(Ju)(s))\,\mathrm{d}s.$$

Note that, for $n \geqslant 1$, Φ_n is decreasing with respect to K, and hence we have $0 \leqslant \Phi_n^2(0) \leqslant \Phi_n(0)$. Let

$$\Omega_1 := \{u \in K;\ 0 \leqslant u(t) \leqslant \Phi_n(0)(t) \text{ for } t \in [0,2\pi]\},$$

then for any $u \in \Omega_1$ we can easily vertify that $\Phi_n(0) \geqslant \Phi_n(u) \geqslant \Phi_n^2(0) \geqslant 0$, this shows that $\Phi_n(\Omega_1) \subset \Omega_1$. In addition, by the definition of Φ_n, it is easily to see that Φ_n is a compact and continuous mapping from Ω_1 to itself. The Schauder fixed point theorem tells us that Φ_n has at least one fixed point $u_n(t)$ in Ω_1.

By the same argument as in Lemma 4, it can be easily shown that there exist two constants $0 < r < R$ such that, for all $n \geqslant 1$,

$$r \leqslant u_n(t) \leqslant R.$$

Now we define the mapping $\Phi : \Omega \to K$ by

$$(\Phi u)(t) := \int_0^{2\pi} G(t,s) f(s,(Ju)(s))\,\mathrm{d}s,$$

where

$$\Omega := \{u \in K;\ r \leqslant u(t) \leqslant R \text{ for } t \in [0,2\pi]\},$$

then $\{u_n\} \subset \Omega$. it is easy to see that Φ is a compact continuous mapping from Ω to K, and so, there is a subsequence of $\{\Phi u_n\}$ which converges to some $u^* \in K$. We relabel the subsequence as original sequence so that $\lim_{n\to\infty} \|\Phi u_n - u^*\| = 0$. Since

$$\begin{aligned}(Ju_n)(t) &= \int_0^t \frac{\mathrm{e}^{\rho(2\pi+x-t)}}{\mathrm{e}^{2\rho\pi}-1} u_n(x)\,\mathrm{d}x + \int_t^{2\pi} \frac{\mathrm{e}^{\rho(x-t)}}{\mathrm{e}^{2\rho\pi}-1} u_n(x)\,\mathrm{d}x \\ &\geqslant \frac{r\mathrm{e}^{-\rho t}}{\mathrm{e}^{2\rho\pi}-1}\left(\int_0^t \mathrm{e}^{\rho(2\pi+x)}\,\mathrm{d}x + \int_t^{2\pi} \mathrm{e}^{\rho x}\,\mathrm{d}x\right) \\ &= \frac{r}{\rho},\end{aligned}$$

further, there exists an $n_0 \geqslant 1$ such that for $n \geqslant n_0$ and $t \in [0,2\pi]$ implies $1/n < r/\rho$, and hence, for $n \geqslant n_0$ we have

$$f_n(t,(Ju_n)(t)) = f(t,(Ju_n)(t)).$$

So, for $n \geqslant n_0$ and $t \in [0, 2\pi]$,

$$\begin{aligned}(\Phi u_n)(t) - u_n(t) &= (\Phi u_n)(t) - (\Phi_n u_n)(t) \\ &= \int_0^{2\pi} G(t,s)[f(s,(Ju_n)(s)) - f_n(s,(Ju_n)(s))]\,\mathrm{d}s \\ &= 0,\end{aligned}$$

i.e. $\lim_{n\to\infty} \|\Phi u_n - u_n\| = 0$. Moreover, we get $\lim_{n\to\infty} \|u_n - u^*\| = 0$, and thus $u^* \in \Omega$, and

$$\Phi u^* = \Phi\left(\lim_{n\to\infty} \Phi u_n\right) = \Phi\left(\lim_{n\to\infty} u_n\right) = \lim_{n\to\infty} \Phi u_n = u^*,$$

this shows that Φ has one fixed point $u^*(t)$ in Ω, and hence $u^*(t)$ is a positive solution to problem (2.6). Since $r \leqslant u^*(t) \leqslant R$ in $[0, 2\pi]$ and problem (2.5) is equivalent to (2.6), $u^*(t)$ is also positive solution to problem (2.5).

Let $y(t) = (Ju^*)(t)$, then it can be easily vertified that $y(t)$ is a positive solution to problem (1.2).

References

[1] J.J. Nieto, Nonlinear second order periodic boundary value problems, J. Math. Anal. Appl. 130 (1988) 22–29.

[2] J.J. Nieto, Nonlinear second order periodic boundary value problems with Carathéodory functions, Anal. Appl. 34 (1989) 111–128.

[3] A. Cabada, J.J. Nieto, Extremal solutions of second order nonlinear periodic boundary value problems, Appl. Math. Comput. 40 (1990) 135–145.

[4] B. Rudolf, Z. Kubacek, Remark on J.J. Nieto's paper: Nonlinear second order periodic boundary value problems, J. Math. Anal. Appl. 146 (1990) 203–206.

[5] A. Cabada, J.J. Nieto, A generalization of the monotone iterative technique for nonlinear second order periodic boundary value problems, J. Math. Anal. 151 (1990) 181–189.

[6] V. Lakshmikantham, J.J. Nieto, Y. Sun, An existence result about periodic boundary value problems of second order differential equations, Appl. Anal. 40 (1990) 1–10.

[7] P. Omari, M. Trombetta, Remark on the lower and upper solutions method for second and third-order periodic boundary value problems, Appl. Math. Comput. 50 (1992) 1–21.

[8] V. Seda, J.J. Nieto, M. Gera, Periodic boundary value problem for nonlinear higher ordinary differential equations, Appl. Math. Comput. 48 (1992) 71–82.

[9] M.X. Wang, A. Cabada, J.J. Nieto, Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions, Ann. Polon. Math. 58 (1993) 221–235.

[10] A. Cabada, The method of lower and upper solutions for second, third, fourth and higher order boundary value problem, J. Math. Anal. Appl. 185 (1994) 302–320.

[11] J.Y. Wang, W.J. Gao, On a nonlinear second order periodic boundary value problems with Carathéodory functions, Ann. Polon. Math. 62 (1995) 283–291.

[12] D.Q. Jiang, J.Y. Wang, A generalized periodic boundary value problem for the one-dimentional p-Laplacian, Ann. Polon. Math. 65 (1997) 265–270.

[13] L.B. Kong, D.Q. Jiang, Multiple positive solutions of a nonlinear fourth order periodic boundary value problem, Ann. Polon. Math. 69 (1998) 265–270.

[14] J.Y. Wang, D.Q. Jiang, A singular nonlinear second-order periodic boundary value problem, J. Tohoku Math. 50 (1998) 203–210.